TWISTED γ-FILTRATION OF A LINEAR ALGEBRAIC GROUP.

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Abstract. In the present notes we introduce and study the twisted γ-filtration on $K_0(G_s)$, where G_s is a split simple linear algebraic group over a field k of characteristic prime to the order of the center of G_s. We apply this filtration to construct torsion elements in the γ-ring of the variety of complete G-flags, where G is an inner form of G_s.

1. Introduction

Let X be a smooth projective variety over a field k. Consider the Grothendieck γ-filtration on $K_0(X)$. It is given by the ideals [6, §2.3] (see also [8, §2])

$$\gamma^i K_0(X) = \langle c_{n_1}(b_1)\cdot \ldots \cdot c_{n_m}(b_m) \mid n_1+\ldots+n_m \geq i,\ b_1,\ldots,b_m \in K_0(X)\rangle,\ i\geq 0$$

generated by products of Chern classes in K_0. Let $\gamma^i(X)$ denote the i-th subsequent quotient and let $\gamma^*(X)=\oplus_{i\geq 0}\gamma^i(X)$ denote the associated graded commutative ring called the γ-ring of X.

The ring $\gamma^*(X)$ was invented by Grothendieck to approximate the topological filtration on K_0 and, hence, the Chow ring $\mathrm{CH}^*(X)$ of algebraic cycles modulo rational equivalence. Indeed, by the Riemann-Roch theorem (see [6, §2]) the i-th Chern class c_i induces an isomorphism with $\mathbb{Q}$-coefficients, i.e. $c_i\colon \gamma^i(X;\mathbb{Q}) \xrightarrow{\simeq} \mathrm{CH}^i(X;\mathbb{Q})$. Moreover, in some cases the ring $\gamma^*(X)$ can be used to compute $\mathrm{CH}^*(X)$, e.g. $\gamma^1(X)=\mathrm{CH}^1(X)$ and there is a surjection $\gamma^2(X)\twoheadrightarrow \mathrm{CH}^2(X)$ (see [7, Ex. 15.3.6]).

In the present notes we provide a uniform lower bound for the torsion part of $\gamma^*(X)$, where $X={}_\xi\mathfrak{B}_s$ is a twisted form of the variety of Borel subgoups $\mathfrak{B}_s$ of a split simple linear algebraic group G_s by means of a cocycle $\xi\in H^1(k,G_s)$. Note that the groups $\gamma^2(X)$ and $\gamma^3(X)$ have been studied for $G_s=PGL_n$ in [8] and for strongly inner forms in [4]. In particular, it was shown in [4, §3,7] that in the strongly inner case the torsion part of $\gamma^2(X)$ determines the Rost invariant.

Our main tool is the twisted γ-filtration on $K_0(G_s)$, where G_s is a split simple linear algebraic group. Roughly speaking, it is defined to be the image (see Definition 4.4) of the γ-filtration on K_0 of the twisted form X under the composition $K_0(X)\to K_0(\mathfrak{B}_s)\to K_0(G_s)$, where the first map is given by the restriction and the second map is induced by taking the quotient.

Let γ^*_ξ denotes the associated graded ring of the twisted γ-filtration. It has the following important properties:

(i) The ring γ^*_ξ can be explicitly computed (see Theorem 4.5). Observe that $\gamma^0_\xi=\mathbb{Z}$, $\gamma^1_\xi=0$ and γ^i_ξ is torsion for $i>1$.
(ii) There is a surjective ring homomorphism $\gamma^*(X)\twoheadrightarrow\gamma^*_\xi$. Hence, γ^*_ξ gives a lower bound for the γ-ring of the twisted form $X={}_\xi\mathfrak{B}_s$.

(iii) The assignment $\xi \mapsto \gamma^*_\xi$ respects the base change and, therefore, is an invariant of a G_s-torsor ξ, moreover, the ring γ^*_ξ can be viewed as a substitute for the γ-ring of the inner group ${}_\xi G_s$.

In the last section we use these properties to construct nontrivial torsion elements in $\gamma^2(X)$ for some twisted flag varieties X (see 5.3 and 5.5). In particular, we establish the connection between the indexes of the Tits algebras of ξ and the order of the special cycle $\theta \in \gamma^2(X)$ constructed in [4].

2. Preliminaries.

In the present section we recall several basic facts concerning linear algebraic groups, characters and the Grothendieck K_0 (see [9, §24], [4, §1B,§6]).

2.1. Let G_s be a split simple linear algebraic group of rank n over a field k. We assume that characteristic of k is prime to the order of the center of G_s. We fix a split maximal torus T and a Borel subgroup B such that $T \subset B \subset G_s$.

Let Λ_r and Λ be the root and the weight lattices of the root system of G_s with respect to $T \subset B$. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a set of simple roots (a basis of Λ_r) and let $\{\omega_1, \ldots, \omega_n\}$ be the respective set of fundamental weights (a basis of Λ), i.e. $\alpha_i^\vee(\omega_j) = \delta_{ij}$. The group of characters T^* of T is an intermediate lattice $\Lambda_r \subset T^* \subset \Lambda$ that determines the isogeny class of G_s. If $T^* = \Lambda$, then the group G_s is simply connected and if $T^* = \Lambda_r$ it is adjoint.

2.2. Let $\mathbb{Z}[T^*]$ be the integral group ring of T^*. Its elements are finite linear combinations $\sum_i a_i e^{\lambda_i}$, $\lambda_i \in T^*$. Let $\mathfrak{B}_s$ denote the variety of Borel subgroups G_s/B of G_s. Consider the characteristic map for K_0 (see [3, §2.8])

$$\mathfrak{c}\colon \mathbb{Z}[T^*] \to K_0(\mathfrak{B}_s)$$

defined by sending e^λ, $\lambda \in T^*$, to the class of the associated line bundle $[\mathcal{L}(\lambda)]$. Observe that the ring $K_0(\mathfrak{B}_s)$ does not depend on the isogeny class of G_s while the group of characters T^* and, hence, the image of $\mathfrak{c}$ does.

Since $K_0(\mathfrak{B}_s)$ is generated by the classes $[\mathcal{L}(\omega_i)]$, $i = 1 \ldots n$, the characteristic map $\mathfrak{c}$ is surjective if G_s is simply connected. If G_s is adjoint, then the image of $\mathfrak{c}$ is generated by the classes $[\mathcal{L}(\alpha_i)]$, where

$$\alpha_i = \sum_j c_{ij}\omega_j \quad \text{and, therefore,} \quad \mathcal{L}(\alpha_i) = \otimes_j \mathcal{L}(\omega_j)^{\otimes c_{ij}},$$

and $c_{ij} = \alpha_i^\vee(\alpha_j)$ are the coefficients of the Cartan matrix of G_s.

2.3. The Weyl group W of G_s acts on weights via simple reflections s_{α_i} as

$$s_{\alpha_i}(\lambda) = \lambda - \alpha_i^\vee(\lambda)\alpha_i, \quad \lambda \in \Lambda.$$

For each element $w \in W$ we define (cf. [13, §2.1]) the weight $\rho_w \in \Lambda$ as

$$\rho_w = \sum_{\{i \in 1\ldots n \mid w^{-1}(\alpha_i) < 0\}} w^{-1}(\omega_i).$$

In particular, for a simple reflection $w = s_{\alpha_j}$ we have

$$\rho_w = \sum_{\{i \in 1\ldots n \mid s_{\alpha_j}(\alpha_i) < 0\}} s_{\alpha_j}(\omega_i) = s_{\alpha_j}(\omega_j) = \omega_j - \alpha_j.$$

Observe that the quotient Λ/Λ_r coincides with the group of characters of the center of the simply connected cover of G_s. Since W acts trivially on Λ/Λ_r, we have

$$\bar{\rho}_w = \sum_{\{i\in 1\ldots n \mid w^{-1}(\alpha_i)<0\}} \bar{\omega}_i \quad \in \Lambda/T^*,$$

where $\bar{\rho}_w$ denotes the class of $\rho_w \in \Lambda$ modulo T^*. In particular, $\bar{\omega}_i = \bar{\rho}_{s_{\alpha_i}}$.

2.4. Let $\mathbb{Z}[\Lambda]^W$ denote the subring of W-invariant elements. Then the integral group ring $\mathbb{Z}[\Lambda]$ is a free $\mathbb{Z}[\Lambda]^W$-module with the basis $\{e^{\rho_w}\}_{w\in W}$ (see [13, Thm.2.2]). Now let $\epsilon\colon \mathbb{Z}[\Lambda] \to \mathbb{Z},\ e^{\lambda} \mapsto 1$ be the augmentation map. By the Chevalley Theorem the kernel of the surjection $\mathfrak{c}$ is generated by elements $x \in \mathbb{Z}[\Lambda]^W$ such that $\epsilon(x) = 0$. Hence, there is an isomorphism

$$\mathbb{Z}[\Lambda] \otimes_{\mathbb{Z}[\Lambda]^W} \mathbb{Z} \simeq \mathbb{Z}[\Lambda]/\ker(\mathfrak{c}) \simeq K_0(\mathfrak{B}_s).$$

So the elements

$$\{g_w = \mathfrak{c}(e^{\rho_w}) = [\mathcal{L}(\rho_w)]\}_{w\in W}$$

form a $\mathbb{Z}$-basis of $K_0(\mathfrak{B}_s)$ called the Steinberg basis.

2.5. Following [14] we associate with each $\chi \in \Lambda/T^*$ and each cocycle $\xi \in Z^1(k, G_s)$ the central simple algebra $A_{\chi,\xi}$ over k called the Tits algebra. This defines a group homomorphism

$$\beta_\xi\colon \Lambda/T^* \to Br(k) \text{ with } \beta_\xi(\chi) = [A_{\chi,\xi}].$$

Let $\mathfrak{B} = {}_\xi\mathfrak{B}_s$ denote the twisted form of the variety of Borel subgroups $\mathfrak{B}_s$ by means of ξ. Consider the restriction map on K_0 over the separable closure k_{sep}

$$res\colon K_0(\mathfrak{B}) \to K_0(\mathfrak{B} \times_k k_{sep}) = K_0(\mathfrak{B}_s),$$

where we identify $K_0(\mathfrak{B} \times_k k_{sep})$ with $K_0(\mathfrak{B}_s)$. By [11, Thm.4.2] the image of the restriction can be identified with the sublattice

$$\langle \imath_w \cdot g_w \rangle_{w\in W},$$

where $g_w = [\mathcal{L}(\rho_w)]$ is an element of the Steinberg basis and $\imath_w = \mathrm{ind}(\beta_\xi(\bar{\rho}_w))$ is the index of the respective Tits algebra. Observe that if G_s is simply connected, then all indexes $\imath_w$ are trivial and the restriction map becomes an isomorphism.

3. The K_0 of a split simple (adjoint) group

In the present section we provide an explicit description of the ring $K_0(G_s)$ in terms of generators and relations for every simple split linear algebraic group G_s. The method to compute $K_0(G_s)$ was known before, however, due to the lack of precise references we provide the computations here.

3.1. **Definition.** Let $\mathfrak{c}\colon \mathbb{Z}[\Lambda] \to K_0(\mathfrak{B}_s)$ be the characteristic map for the simply connected cover of G_s. We define the ring $\mathfrak{G}_s$ to be the quotient

$$\mathfrak{G}_s := \mathbb{Z}[\Lambda/T^*]/\overline{(\ker \mathfrak{c})}$$

and the surjective ring homomorphism q to be the composite

$$q\colon K_0(\mathfrak{B}_s) \xrightarrow[\simeq]{\mathfrak{c}^{-1}} \mathbb{Z}[\Lambda]/(\ker \mathfrak{c}) \twoheadrightarrow \mathbb{Z}[\Lambda/T^*]/\overline{(\ker c)} = \mathfrak{G}_s.$$

Observe that if G_s is simply connected, then $\mathfrak{G}_s = \mathbb{Z}$.

3.2. **Remark.** By [10, Cor.33] applied to $X = G_s$ and to the simply-connected cover $G = \hat{G}_s$ of G_s, there is an isomorphism

$$K_0(G_s) \simeq \mathbb{Z} \otimes_{R(\hat{G}_s)} K_0(\hat{G}_s, G_s),$$

where $R(\hat{G}_s) \simeq \mathbb{Z}[\Lambda]^W$ is the representation ring. By [10, Cor.5] applied to $G = \hat{G}_s$, $X = Spec\, k$ and $G/H = G_s$ there is an isomorphism

$$K_0(\hat{G}_s, G_s) \simeq R(H),$$

where $R(H) \simeq \mathbb{Z}[\Lambda/T^*]$ is the representation ring. Therefore,

$$K_0(G_s) \simeq \mathbb{Z} \otimes_{\mathbb{Z}[\Lambda]^W} \mathbb{Z}[\Lambda/T^*] \simeq \mathfrak{G}_s.$$

3.3. **Lemma.** *The ideal* $\overline{(\ker \mathfrak{c})} \subset \mathbb{Z}[\Lambda/T^*]$ *is generated by the elements*

$$d_i(1 - e^{\bar{\omega}_i}),\ i = 1 \dots n,$$

where d_i *is the dimension of the* i*-th fundamental representation.*

Proof. By the Chevalley Theorem the subring of invariants $\mathbb{Z}[\Lambda]^W$ can be identified with the polynomial ring $\mathbb{Z}[\rho_1, \dots, \rho_n]$, where ρ_i is the i-th fundamental representation, i.e.

$$\rho_i = \sum_{\lambda \in W(\omega_i)} e^\lambda$$

(here $W(\omega_i)$ denotes the W-orbit of the fundamental weight ω_i).

Since $d_i = \epsilon(\rho_i)$, $\ker \mathfrak{c} = (d_1 - \rho_1, \dots, d_n - \rho_n)$. To finish the proof observe that $\overline{(d_i - \rho_i)} = d_i(1 - e^{\bar{\omega}_i})$. $\square$

3.4. **Remark.** Observe that by definition and 3.3 we have $\mathfrak{G}_s \otimes \mathbb{Q} \simeq \mathbb{Q}$.

3.5**.** In the following examples we compute the ring $\mathfrak{G}_s \simeq K_0(G_s)$ for every simple split linear algebraic group G_s (we refer to [9, §24] for the description of Λ/T^* and to [1, Ch.8, Table 2] for the dimensions of fundamental representations).

Λ/T^*	G_s, $m \geq 1$	Example
$\mathbb{Z}/m\mathbb{Z}$, $m \geq 2$	SL_{n+1}/μ_m	(3.6)
$\mathbb{Z}/2\mathbb{Z}$	O^+_{m+4}, PSp_{2m+2}, $HSpin_{4m+4}$, E_7^{ad}	(3.7)
$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	PGO^+_{4m+4}	(3.8)
$\mathbb{Z}/3\mathbb{Z}$	E_6^{ad}	(3.9)
$\mathbb{Z}/4\mathbb{Z}$	PGO^+_{4m+2}	(3.10)

3.6. **Example.** Consider the case $G_s = SL_{n+1}/\mu_m$, $m \geq 2$. The group G_s has type A_n and $\Lambda/T^* = \langle\sigma\rangle$ is cyclic of order m. The quotient map $\Lambda/\Lambda_r \to \Lambda/T^*$ sends $\bar{\omega}_i \in \Lambda/\Lambda_r$, $i = 1 \dots n$ to $(i \bmod m)\sigma \in \Lambda/T^*$.

By Definition 3.1 and Lemma 3.3 we have

$$\mathfrak{G}_s \simeq \mathbb{Z}[y]/(1 - (1-y)^m, d_1 y, \dots, d_{m-1}y^{m-1}),$$

where $y = (1 - e^\sigma)$ and $d_j = gcd\{\binom{n+1}{i} \mid i \equiv j \mod m,\ i = 1 \dots n\}$.

In particular, for $G_s = SL_p/\mu_p = PGL_p$, where p is a prime, we obtain

$$\mathfrak{G}_s \simeq \mathbb{Z}[y]/(\tbinom{p}{1}y, \tbinom{p}{2}y^2, \dots, \tbinom{p}{p-1}y^{p-1}, y^p).$$

3.7. **Example.** Assume that $\Lambda/T^* = \langle\sigma\rangle$ has order 2. Then

$$\mathfrak{G}_s \simeq \mathbb{Z}[y]/(y^2-2y, dy),$$

where $y=(1-e^\sigma)$ and d is the g.c.d. of dimensions of representations corresponding to ω_i with $\bar\omega_i=\sigma$. The integer d can be determined as follows:

B_n: We have $\Lambda/\Lambda_r = \{0,\bar\omega_n\} \simeq \mathbb{Z}/2\mathbb{Z}$ which corresponds to the adjoint group $G_s = O^+_{2n+1}$. Since $\bar\omega_i=0$ for each $i\neq n$, d coincides with the dimension of ω_n that is 2^n.

C_n: We have $\Lambda/\Lambda_r = \{0,\sigma=\bar\omega_1=\bar\omega_3=\ldots\} \simeq \mathbb{Z}/2\mathbb{Z}$ that is $G_s = PSp_{2n}$. Since $\bar\omega_i=0$ for even i, d is the g.c.d. of dimensions of $\omega_1,\omega_3,\ldots$, i.e.

$$d = gcd\big(2n, \tbinom{2n}{3}-\tbinom{2n}{1}, \tbinom{2n}{5}-\tbinom{2n}{3},\ldots\big).$$

D_n: If n is odd, then $\Lambda/\Lambda_r = \{0,\bar\omega_{n-1},\bar\omega_1,\bar\omega_n\}\simeq \mathbb{Z}/4\mathbb{Z}$, where $\bar\omega_1 = 2\bar\omega_{n-1}=2\bar\omega_n$. Therefore, $\Lambda/T^*\simeq\mathbb{Z}/2\mathbb{Z}$ if it is a quotient of Λ/Λ_r modulo the subgroup $\{0,\bar\omega_1\}$. In this case $\Lambda/T^* = \{0,\sigma=\bar\omega_{n-1}=\bar\omega_n\}$ which corresponds to the special orthogonal group $G_s = O^+_{2n}$. Since $\bar\omega_s = s\bar\omega_1$ for $2\le s\le n-2$ and $\bar\omega_1=0$ in Λ/T^*, d is the g.c.d. of the dimensions of ω_{n-1} and ω_n that is 2^{n-1}.

If n is even, then $\Lambda/\Lambda_r = \{0,\bar\omega_{n-1}\}\oplus\{0,\bar\omega_n\}\simeq \mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$, where $\bar\omega_1=\bar\omega_{n-1}+\bar\omega_n$. In this case, we have two cases for Λ/T^*:

(1) It is the quotient of Λ/Λ_r modulo the diagonal subgroup $\{0,\bar\omega_{n-1}+\bar\omega_n\}$. Then $\Lambda/T^*=\{0,\sigma=\bar\omega_{n-1}=\bar\omega_n\}$, $G_s=O^+_{2n}$ and d is the same as in the odd case, i.e. $d=2^{n-1}$.
(2) It is the quotient modulo one of the factors, e.g. $\Lambda/T^*=\{0,\sigma=\bar\omega_{n-1}\}$, where $\bar\omega_n=0$. This corresponds to the half-spin group $G_s = HSpin_{2n}$. We have $\bar\omega_1=\bar\omega_3=\ldots=\bar\omega_{n-1}$ and $\bar\omega_i=0$ if i is even.
Therefore, $d = gcd\big(2n,\tbinom{2n}{3},\ldots,\tbinom{2n}{n-3},2^{n-1}\big)$ which implies that $d=2^{v_2(n)+1}$, where $v_2(n)$ denotes the 2-adic valuation of n.

E_7: We have $\Lambda/\Lambda_r=\{0,\sigma=\bar\omega_7=\bar\omega_5=\bar\omega_2\}\simeq\mathbb{Z}/2\mathbb{Z}$ with $\bar\omega_1=\bar\omega_3=\bar\omega_4=\bar\omega_6=0$. Therefore, $d=gcd(56,\tbinom{56}{3},912)=8$.

3.8. **Example.** Assume that $\Lambda/T^*=\langle\sigma_1\rangle\oplus\langle\sigma_2\rangle$, where σ_1 and σ_2 are of order 2. In this case $G_s = PGO^+_{2n}$ is an adjoint group ($T^*=\Lambda_r$) of type D_n with n even. We have $\sigma_1=\bar\omega_{n-1}$ and $\sigma_2=\bar\omega_n$, $\bar\omega_s=s\bar\omega_1$, $2\le s\le n-2$, $2\bar\omega_1=0$ and $\bar\omega_1=\bar\omega_{n-1}+\bar\omega_n$. Then

$$\mathfrak{G}_s\simeq\mathbb{Z}[y_1,y_2]/(y_1^2-2y_1,y_2^2-2y_2,d_1y_1,d_2y_2,d(y_1+y_2-y_1y_2)),$$

where $y_1=(1-e^{\sigma_1})$, $y_2=(1-e^{\sigma_2})$; d_1 (resp. d_2) is the g.c.d. of dimensions of ω_i with $\bar\omega_i=\bar\omega_{n-1}$ (resp. $\bar\omega_i=\bar\omega_n$) that is $d_1=d_2=2^{n-1}$; and d is the g.c.d. of dimensions of $\omega_1,\omega_3,\ldots,\omega_{n-3}$ that is $d=gcd\big(2n,\tbinom{2n}{3},\ldots\tbinom{2n}{n-3}\big)$.

In particular, for $G_s=PGO^+_8$ we obtain

$$\mathfrak{G}_s\simeq\mathbb{Z}[y_1,y_2]/(y_1^2-2y_1,y_2^2-2y_2,8y_1,8y_2,8y_1y_2).$$

3.9. **Example.** Assume that $\Lambda/T^*=\langle\sigma\rangle$ has order 3. Then

$$\mathfrak{G}_s\simeq\mathbb{Z}[y]/(y^3-3y^2+3y,d_1y,d_2y^2),$$

where $y=(1-e^\sigma)$ and d_1 (resp. d_2) is the greatest common divisor of dimensions of fundamental representations ω_i, $i=1\ldots n$ such that $\bar\omega_i=\sigma$ (resp. $\bar\omega_i=2\sigma$).

For the adjoint group of type E_6 we have $\Lambda/\Lambda_r = \{0, \sigma = \bar\omega_1 = \bar\omega_5, 2\sigma = \bar\omega_2 = \bar\omega_6\}$ with $\bar\omega_2 = \bar\omega_4 = 0$. Therefore, $d_1 = d_2 = gcd(27, \binom{27}{2}) = 27$.

3.10. **Example.** Assume that $\Lambda/T^* = \langle\sigma\rangle$ has order 4. Then

$$\mathfrak{G}_s \simeq \mathbb{Z}[y]/(y^4 - 4y^3 + 6y^2 - 4y, d_1 y, d_2 y^2, d_3 y^3),$$

where $y = (1 - e^{\sigma})$. For the group PGO^+_{2n} where n is odd we have $\sigma = \bar\omega_{n-1}$, $2\sigma = \bar\omega_1$ and $3\sigma = \bar\omega_n$. Therefore, $d_1 = d_3 = 2^{n-1}$ and $d_2 = gcd(\binom{2n}{1}, \binom{2n}{3}, \ldots, \binom{2n}{n-2})$.

4. The twisted γ-filtration.

In the present section we introduce and study the twisted γ-filtration.

4.1. Let $\gamma = \ker\epsilon$ denote the augmentation ideal in $\mathbb{Z}[\Lambda]$. It is generated by the differences

$$\langle (1 - e^{-\lambda}),\ \lambda \in \Lambda\rangle.$$

Consider the γ-adic filtration on $\mathbb{Z}[\Lambda]$

$$\mathbb{Z}[\Lambda] = \gamma^0 \supseteq \gamma \supseteq \gamma^2 \supseteq \ldots$$

The i-th power γ^i is generated by products of at least i differences.

4.2. **Definition.** We define the filtration on $K_0(\mathfrak{B}_s)$ (resp. on $\mathfrak{G}_s$) to be the image of the γ-adic filtration on $\mathbb{Z}[\Lambda]$ via $\mathfrak{c}$ (resp. via q), i.e.

$$\gamma^i K_0(\mathfrak{B}_s) := \mathfrak{c}(\gamma^i) \text{ and } \gamma^i\mathfrak{G}_s := q(\gamma^i K_0(\mathfrak{B}_s)),\ i \geq 0.$$

So that we have a commutative diagram of surjective group homomorphisms

$$\begin{tikzcd} \gamma^i \arrow[r, "\mathfrak{c}"] \arrow[dr] & \gamma^i K_0(\mathfrak{B}_s) \arrow[d, "q"] \\ & \gamma^i\mathfrak{G}_s \end{tikzcd}$$

4.3. **Lemma.** *The γ-filtration on $K_0(\mathfrak{B}_s)$ coincides with the filtration introduced in Definition 4.2.*

Proof. Since $K_0(\mathfrak{B}_s)$ is generated by the classes of line bundles,

$$\gamma^i K_0(\mathfrak{B}_s) = \langle c_1([\mathcal{L}_1]) \cdot \ldots \cdot c_1([\mathcal{L}_m]) \mid m \geq i,\ \mathcal{L}_j \in K_0(\mathfrak{B}_s)\rangle.$$

Moreover, each line bundle $\mathcal{L}$ is the associated bundle $\mathcal{L} = \mathcal{L}(\lambda)$ for some character $\lambda \in \Lambda$. Therefore, $c_1([\mathcal{L}]) = 1 - [\mathcal{L}^\vee] = \mathfrak{c}(1 - e^{-\lambda})$ (see [3, §2.8]). $\square$

4.4. **Definition.** Given a G_s-torsor $\xi \in H^1(k, G_s)$ and the respective twisted form $\mathfrak{B} = {}_\xi\mathfrak{B}_s$ we define the twisted filtration on $\mathfrak{G}_s$ to be the image of the γ-filtration on $K_0(\mathfrak{B})$ via the composite $res \circ q$, i.e.

$$\gamma^i_\xi \mathfrak{G}_s := q(res(\gamma^i K_0(\mathfrak{B}))),\ i \geq 0.$$

Let $\gamma^{i/i+1}_\xi\mathfrak{G}_s = \gamma^i_\xi\mathfrak{G}_s/\gamma^{i+1}_\xi\mathfrak{G}_s$. The associated graded ring $\bigoplus_{i\geq 0}\gamma^{i/i+1}_\xi\mathfrak{G}_s$ will be called the γ-invariant of the torsor ξ and will be denoted simply as γ^*_ξ.

Note that the Chern classes commute with restrictions, therefore the restriction map $res\colon \gamma^i K_0(\mathfrak{B}) \to \gamma^i K_0(\mathfrak{B}_s)$ is well-defined. By definition there is a surjective ring homomorphism

$$\gamma^*(\mathfrak{B}) \twoheadrightarrow \gamma^*_\xi.$$

4.5. **Theorem.** *The twisted filtration $\gamma^i_\xi \mathfrak{G}_s$ can be computed as follows:*

$$\gamma^i_\xi \mathfrak{G}_s = \langle \prod_{j=1}^m \binom{\mathrm{ind}(\beta_\xi(\bar\rho_{w_j}))}{n_j}(1-e^{\bar\rho_{w_j}})^{n_j} \mid n_1+\ldots+n_m \geq i,\ w_j \in W\rangle.$$

Proof. Since the Chern classes commute with restrictions, the image of the restriction $res\colon \gamma^i K_0(\mathfrak{B}) \to \gamma^i K_0(\mathfrak{B}_s)$ is generated by the products

$$\langle c_{n_1}(\imath_{w_1} g_{w_1})\cdot \ldots \cdot c_{n_m}(\imath_{w_m} g_{w_m}) \mid n_1+\ldots+n_m \geq i,\ w_1,\ldots,w_m \in W\rangle,$$

where $\{\imath_{w_j}\}$ are the indexes of the respective Tits algebras from 2.5. Applying the Whitney formula for the Chern classes [7, §3.2] we obtain

$$c_j(\imath_w g_w) = \binom{\imath_w}{j} c_1(g_w)^j.$$

Therefore, $q(\binom{\imath_w}{j}c_1(g_w)^j) = \binom{\imath_w}{j}(1-e^{-\bar\rho_w})^j$, where $\imath_w = \mathrm{ind}(\beta_\xi(\bar\rho_w))$. $\square$

4.6. **Example.** Since $\gamma^0(X) \simeq \mathbb{Z}$ and $\gamma^1(X) = Pic(X)$ is torsion free for every smooth projective X, we obtain that $\gamma^0_\xi \simeq \mathbb{Z}$ and $\gamma^1_\xi = 0$ for any ξ.

4.7. **Example** (Strongly-inner case)**.** If $\beta_\xi = 0$, then $\binom{\imath_{w_j}}{n_j} = 1$ and $\gamma^i_\xi \mathfrak{G}_s = \gamma^i \mathfrak{G}_s$.

4.8. **Example** ($\mathbb{Z}/2\mathbb{Z}$-case)**.** As in 3.7 assume that $\Lambda/T^* = \langle\sigma\rangle$ has order 2 and $\beta_\xi \neq 0$. Then there is only one non-split Tits algebra $A = A_{\sigma,\xi}$ and it has exponent 2. Let $\mathsf{i}_A = v_2(\mathrm{ind}(A))$ denote the 2-adic valuation of the index of A. By definition we have

$$\gamma^i_\xi \mathfrak{G}_s = \langle \tbinom{2^{\mathsf{i}_A}}{n_1}\ldots\tbinom{2^{\mathsf{i}_A}}{n_m} 2^{n_1+\ldots+n_m-1} y \mid n_1+\ldots+n_m \geq i\rangle$$

in $\mathbb{Z}[y]/(y^2-2y, dy)$, where $y = 1-e^\sigma$ and d is given in 3.7. Observe that modulo the relation $y^2 = 2y$ these ideals are generated by (for $j \geq 1$)

$$\begin{array}{ll}
\gamma^{2j-1}_\xi \mathfrak{G}_s = \gamma^{2j}_\xi \mathfrak{G}_s = \langle 2^{2j-1}y\rangle & \text{if } \mathsf{i}_A = 1;\\
\gamma^{4j-3}_\xi \mathfrak{G}_s = \gamma^{4j-2}_\xi \mathfrak{G}_s = \langle 2^{4j-2}y\rangle,\ \gamma^{4j-1}_\xi \mathfrak{G}_s = \gamma^{4j}_\xi \mathfrak{G}_s = \langle 2^{4j-1}y\rangle & \text{if } \mathsf{i}_A = 2;\\
\gamma^{1}_\xi \mathfrak{G}_s = \gamma^{2}_\xi \mathfrak{G}_s = \langle 2^{\mathsf{i}_A}y\rangle,\ \gamma^{3}_\xi \mathfrak{G}_s = \gamma^{4}_\xi \mathfrak{G}_s = \langle 2^{\mathsf{i}_A+1}y\rangle, \gamma^{5}_\xi \mathfrak{G}_s = \langle 2^{\mathsf{i}_A+4}y\rangle\ldots & \text{if } \mathsf{i}_A > 2.
\end{array}$$

Taking these generators modulo the relation $dy = 0$ we obtain the following formulas for the second quotient γ^2_ξ:

$$\text{if } \mathsf{i}_A = 1, \text{ then } \gamma^2_\xi = \begin{cases} 0 & \text{if } v_2(d) \leq 1 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } v_2(d) = 2 \\ \mathbb{Z}/4\mathbb{Z} & \text{if } v_2(d) \geq 3 \end{cases}$$

$$\text{if } \mathsf{i}_A > 1, \text{ then } \gamma^2_\xi = \begin{cases} 0 & \text{if } v_2(d) \leq \mathsf{i}_A \\ \mathbb{Z}/2\mathbb{Z} & \text{if } v_2(d) > \mathsf{i}_A \end{cases}$$

4.9. **Example** ($\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$-case)**.** As in 3.8 assume that $\Lambda/T^* = \langle\sigma_1\rangle \oplus \langle\sigma_2\rangle$, where σ_1, σ_2 have order 2. This is the case for the adjoint group PGO^+_{2n} where n is even. Assume that $n = 4$ which corresponds to the group of type D_4, i.e. PGO^+_8. Let C^+ and C^- denote the Tits algebras corresponding to the generators $\sigma_1 = \bar\omega_3$ and $\sigma_2 = \bar\omega_4$. Let A denote the Tits algebra corresponding to the sum $\sigma_1+\sigma_2$. Note that $C^+ \times C^-$ is the even part of the Clifford algebra of the algebra with involution A and $[A] = [C^+ \otimes C^-]$ in $Br(k)$.

By definition we have in $\mathbb{Z}[y_1, y_2]$

$$\gamma^i_\xi \mathfrak{G}_s = \langle \tbinom{ind\, C_+}{n_1} y_1^{n_1} \cdot \tbinom{ind\, C_-}{n_2} y_2^{n_2} \cdot \tbinom{ind\, A}{n_3} (y_1 + y_2 - y_1 y_2)^{n_3} \mid n_1 + n_2 + n_3 \geq i \rangle.$$

Modulo the relations $(y_1^2 - 2y_1, y_2^2 - 2y_2, 8y_1, 8y_2, 8y_1y_2)$ we obtain that

$$\gamma^2_\xi \mathfrak{G}_s \simeq \frac{(ind\, C_+)\mathbb{Z}}{8\mathbb{Z}} \oplus \frac{(ind\, C_-)\mathbb{Z}}{8\mathbb{Z}} \oplus \frac{(ind\, A)\mathbb{Z}}{8\mathbb{Z}}$$

5. Torsion in the γ-filtration.

In the present section we show how the twisted γ-filtration can be used to construct nontrivial torsion elements in the γ-ring of the twisted form $\mathfrak{B}$ of a variety of Borel subgroups.

5.1. For simplicity we consider only the case of G_s (see Examples 3.7 and 4.8) with $\Lambda/T^* = \langle \sigma \rangle$ of order 2. Let d denote the g.c.d. of dimensions of fundamental representations corresponding to σ.

Given a G_s-torsor $\xi \in H^1(k, G_s)$ let i_A denote the 2-adic valuation of the index of the Tits algebra $A = A_{\sigma,\xi}$. Let $\mathfrak{B} = {}_\xi\mathfrak{B}_s$ denote the twisted form of the variety of Borel subgroups of G_s by means of ξ. Consider the respective twisted filtration $\gamma^i_\xi \mathfrak{G}_s$ on $\mathfrak{G}_s$.

5.2. **Proposition.** *Assume that $v_2(d) > \mathsf{i}_A \geq 3$. Then for each $\lambda \in \Lambda$ such that $\bar{\lambda} = \sigma$ there exists a non-trivial torsion element of order 2 in $\gamma^2(\mathfrak{B})$. Moreover, its image in $\gamma^2_\xi = \mathbb{Z}/2$ (via q) is non-trivial and in $\gamma^2(\mathfrak{B}_s)$ (via res) is trivial.*

Proof. The proof of this result was inspired by the proof of [8, Prop.4.13].

Let $g = [\mathcal{L}(\lambda)]$ denote the class of the associated line bundle. Using the formula for the first Chern class of a tensor product of line bundles for K_0 we obtain

$$c_1(g)^2 = 2c_1(g) - c_1(g^2).$$

Hence,

$$c_1(g)^4 = (2c_1(g) - c_1(g^2))^2 = 4c_1(g)^2 - 4c_1(g)c_1(g^2) + c_1(g^2)^2.$$

Therefore,

$$\eta = 4c_1(g)^3 - c_1(g)^4 = 4c_1(g)^2 - c_1(g^2)^2 \in \gamma^3 K_0(\mathfrak{B}_s).$$

We claim that the class of $2^{\mathsf{i}_A - 3}\eta$ gives the desired torsion element.

Indeed, $c_1(g^2) = c_1([\mathcal{L}(2\lambda)])$. Since $2\lambda \in T^*$, $[\mathcal{L}(2\lambda)] \in \mathfrak{c}(T^*)$ and, therefore, by [5, Cor.3.1] $c_1(g^2) \in \gamma^1 K_0(\mathfrak{B})$. Moreover, we have $2^{\mathsf{i}_A - 1} c_1(g)^2 = c_2(2^{\mathsf{i}_A} g)$, where $2^{\mathsf{i}_A} g \in K_0(\mathfrak{B})$. Hence, $2^{\mathsf{i}_A - 1} c_1(g)^2 \in \gamma^2 K_0(\mathfrak{B})$. Combining these together we obtain that $2^{\mathsf{i}_A - 3}\eta \in \gamma^2 K_0(\mathfrak{B})$.

Now since $2^{\mathsf{i}_A - 3}\eta \in \gamma^2 K_0(\mathfrak{B})$ its image in $\gamma^2_\xi \mathfrak{G}_s$ can be computed as

$$q(2^{\mathsf{i}_A - 3}\eta) = 2^{\mathsf{i}_A - 3} q(\eta) = 2^{\mathsf{i}_A - 1} q(c_1(g)^2) = 2^{\mathsf{i}_A - 1}(1 - e^{-\sigma})^2 = 2^{\mathsf{i}_A} y.$$

But $q(2^{\mathsf{i}_A - 3}\eta) \notin \gamma^3_\xi \mathfrak{G}_s = \langle 2^{\mathsf{i}_A + 1} y \rangle$. Therefore, $2^{\mathsf{i}_A - 3}\eta \notin \gamma^3 K_0(\mathfrak{B})$.

From the other hand side $2^{\mathsf{i}_A - 2}\eta = 2^{\mathsf{i}_A} c_1(g)^3 + 2^{\mathsf{i}_A - 2} c_1(g)^4$ is in $\gamma^3 K_0(\mathfrak{B})$. So the class of $2^{\mathsf{i}_A - 3}\eta$ gives the desired torsion element of order 2. □

5.3. **Example.** Let $G_s = HSpin_{2n}$ be a half-spin group of rank $n \geq 4$. So G_s is of type D_n, where n is even, $\Lambda/T^* = \langle \sigma = \bar{\omega}_1 \rangle \simeq \mathbb{Z}/2\mathbb{Z}$ and according to Example 3.7 we have $d = 2^{v_2(n)+1}$. Let $\xi \in H^1(k, G_s)$ be a non-trivial torsor. Then there is only one Tits algebra $A = A_{\sigma,\xi}$; it has exponent 2 and index 2^{i_A} such that $\mathsf{i}_A \leq v_2(n)+1$.

Recall that each such torsor corresponds to an algebra with orthogonal involution (A,δ) with trivial discriminant and trivial component of the Clifford algebra. The respective twisted form $\mathfrak{B} = {}_\xi\mathfrak{B}_s$ then corresponds to the variety of Borel subgroups of the group $PGO^+(A,\delta)$.

Applying the proposition to this case we obtain that for any such algebra (A,δ) where $8 \mid \mathrm{ind}(A)$ and A is non-division, there exists a non-trivial torsion element of order 2 in $\gamma^2(\mathfrak{B})$ that vanishes over a splitting field of (A,δ).

5.4. **Lemma.** *The γ-filtration on $K_0(\mathfrak{B}_s)$ is generated by the first Chern classes $c_1([\mathcal{L}(\omega_i)])$, $i = 1\dots n$, i.e.*

$$\gamma^i K_0(\mathfrak{B}_s) = \langle \prod_{j\in 1\dots n} c_1([\mathcal{L}(\omega_j)]) \mid \textit{the number of elements in the product } \geq i\rangle.$$

In particular, the second quotient $\gamma^2(\mathfrak{B}_s)$ is additively generated by the products

$$\gamma^2(\mathfrak{B}_s) = \langle c_1([\mathcal{L}(\omega_i)])c_1([\mathcal{L}(\omega_j)]) \mid i,j \in 1\dots n\rangle.$$

Proof. Each $b \in K_0(\mathfrak{B}_s)$ can be written as a linear combination $b = \sum_{w\in W} a_w g_w$. Therefore, any Chern class of b can be expressed in terms of $c_1(g_w)$.

Each ρ_w can be written uniquely as a linear combination of fundamental weights $\{\omega_1,\dots,\omega_n\}$. Therefore, by the formula for the Chern class of the tensor product of line bundles [2, 8.2], each $c_1(g_w)$ can be expressed in terms of $c_1([\mathcal{L}(\omega_i)])$. □

5.5. **Example.** Let G_s be an adjoint group of type E_7 and let $\xi \in H^1(k,G_s)$ be a non-trivial G_s-torsor. Then there is only one nonsplit Tits algebra $A = A_{\sigma,\xi}$ of exponent 2 and $\mathfrak{i}_A \leq 3$. Let $\mathfrak{B} = {}_\xi\mathfrak{B}_s$ be the respective twisted flag variety.

By Lemma 5.4 any element of $\gamma^2(\mathfrak{B})$ can be written as

$$x = \sum_{ij} a_{ij} c_1([\mathcal{L}(\omega_i)])c_1([\mathcal{L}(\omega_j)]) \in \gamma^2(\mathfrak{B})$$

for certain coefficients $a_{ij} \in \mathbb{Z}$. Since $\sigma = \bar\omega_7 = \bar\omega_5 = \bar\omega_2$ and $\bar\omega_1 = \bar\omega_3 = \bar\omega_4 = \bar\omega_6 = 0$, we obtain that

$$q(x) = C\cdot 2y \in \gamma^2_\xi, \text{ where } C = a_{25}+a_{27}+a_{57}+a_{22}+a_{55}+a_{77}.$$

Therefore, $q(x) \neq 0$ in γ^2_ξ if and only if $4 \nmid C$ and $\mathfrak{i}_A \leq 2$.

Consider the class $\mathfrak{c}(\theta) \in \gamma^2 K_0(\mathfrak{B}_s)$ of the special cycle θ constructed in [4, Def.3.3]. Note that the image of θ in $CH^2(\mathfrak{B})$ can be viewed as a generalization of the Rost invariant for split adjoint groups (see Remark 5.7).

If $\mathfrak{i}_A = 1$, then by [4, Prop.6.5] we know that $\mathfrak{c}(\theta) \in \gamma^2(\mathfrak{B})$ is a non-trivial torsion element. If $\mathfrak{i}_A = 2$, then following the proof of [4, Prop.6.5] we obtain that $2\mathfrak{c}(\theta) \in \gamma^2(\mathfrak{B})$.

We claim that if $\mathfrak{i}_A \leq 2$, then $x = 2\mathfrak{c}(\theta)$ is non-trivial. Indeed, in this case $4 \nmid C = a_{22}+a_{55}+a_{77} = 6$, therefore, we have $q(x)\neq 0$, and $x \neq 0$ in $\gamma^2(\mathfrak{B})$. In particular, this shows that for $\mathfrak{i}_A = 1$ the order of the special cycle θ in $\gamma^2(\mathfrak{B})$ is divisible by 4.

5.6. **Example.** Let $\xi \in H^1(k, PGO_8^+)$. Applying the same arguments as in 5.5 to Example 4.9 we obtain that if $ind(A), ind(C_+), ind(C_-) \leq 4$, then $2\mathfrak{c}(\theta) \in \gamma^2(\mathfrak{B})$ is non-trivial.

We finish by the following remark that provides another motivation for the study of the torsion part of $\gamma^*(\mathfrak{B})$

5.7. **Remark.** Recall that by the Riemann-Roch theorem the second Chern class induces a surjection $c_2 \colon Tors\,\gamma^2(\mathfrak{B}) \twoheadrightarrow Tors\,\mathrm{CH}^2(\mathfrak{B})$ [8, Cor.2.15], where the latter group is isomorphic to the cohomology quotient [12, Thm.2.1]

$$\frac{\ker(H^3(k,\mathbb{Q}/\mathbb{Z}(2)) \to H^3(k(\mathfrak{B}),\mathbb{Q}/\mathbb{Z}(2))}{\oplus_{\chi\in\Lambda/\Lambda_r} N_{k_\chi/k}(k_\chi^* \cup \beta_\xi(\chi))},$$

where k_χ denotes the fixed subfield of χ. Therefore, the group $Tors\,\gamma^2(\mathfrak{B})$ can be viewed as an upper bound for the group of cohomological invariants of G_s-torsors in degree 3.

Acknowledgments. I am grateful to Alexander Merkurjev for pointing out Remark 3.2. This work was supported by NSERC Discovery 385795-2010 and Accelerator Supplement 396100-2010 grants.

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